Chaos and the Chua Circuit

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Why the Chua Circuit?

In the early 80s, the only known autonomous ODE systems that were considered chaotic were the Lorenz Equations and the Rössler Equations.

Lorenz Equations

$$x' = -\alpha(x - y)$$
$$y' = \beta x - y - xz$$
$$z' = xy - \gamma z$$

Depended on round-off errors in initial conditions. Was there a physical system that could demonstrate chaos?

Chaos

Chaos occurs with three-dimensional systems of differential equations with some nonlinearity.

Steady-state behavior is oscillatory (similar to a limit cycle) but is aperiodic.

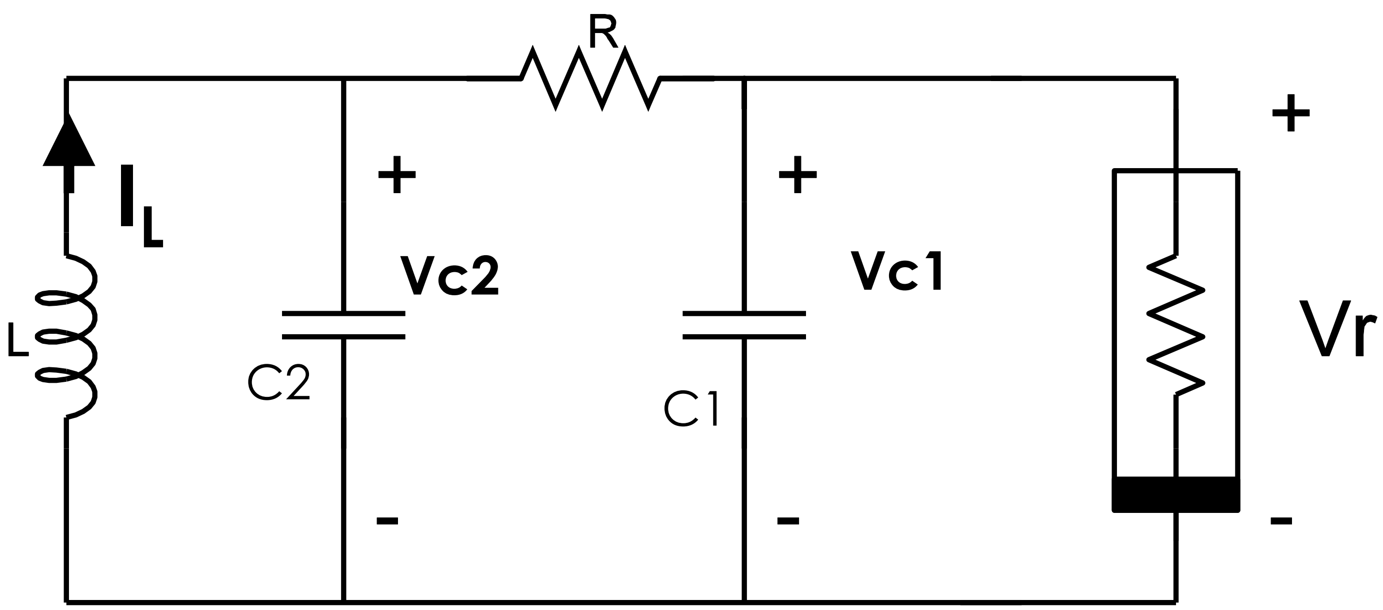
Steady-state behavior is bounded.

Solutions are extremely sensitive to initial conditions.

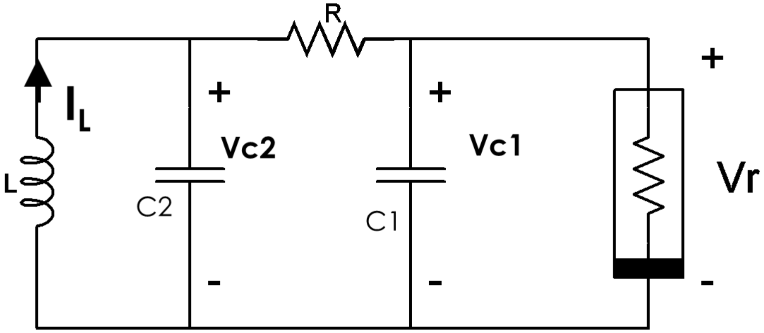
Dr. Leon O. Chua

In 1983, Dr. Leon Chua gave a design idea to Matsumoto to build a circuit to produce a chaotic system

Chua Circuit



Chua Circuit



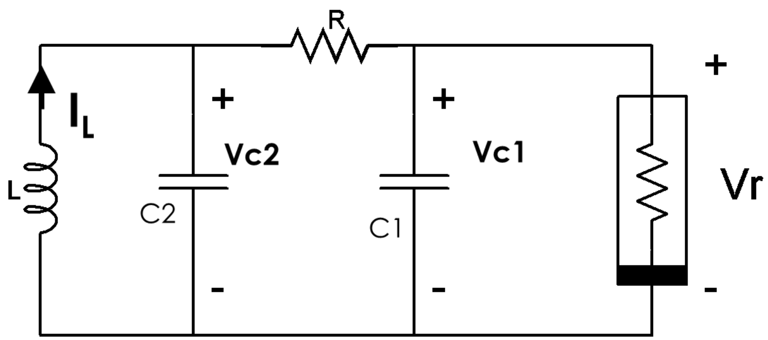
$$C_1 \frac{dv_{c1}}{dt} = \frac{v_{c2} - v_{c1}}{R} - g(v_{c1})$$
$$C_2 \frac{dv_{c2}}{dt} = \frac{v_{c1} - v_{c2}}{R} + i_L$$
$$L \frac{di_L}{dt} = -\mathrm{v}_{c2}$$

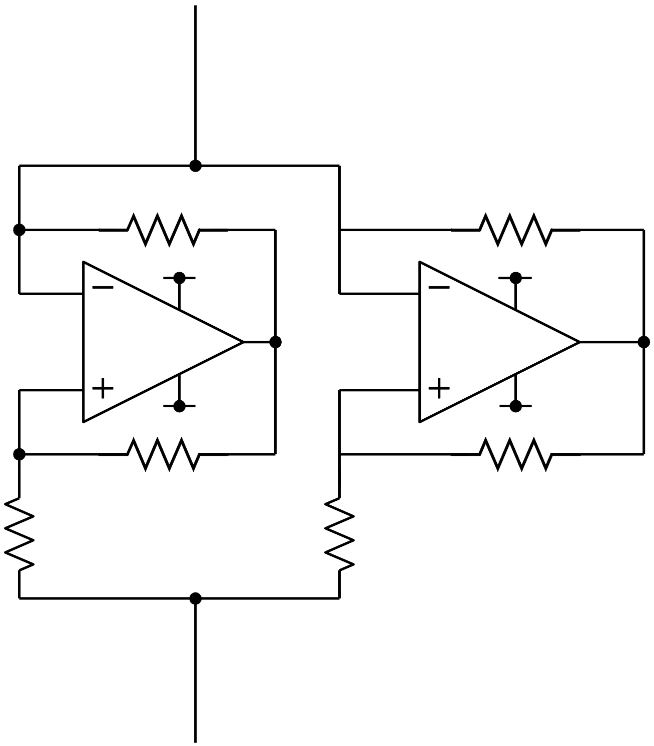
Equations created using Kirchhoff's First and Second Laws

$$\boldsymbol{x' = a(y - \phi(x))}$$
$$\boldsymbol{y' = x - y - z}$$
$$\boldsymbol{z' = -by}$$

Simplified, dimensionless form
With a, b positive parameters

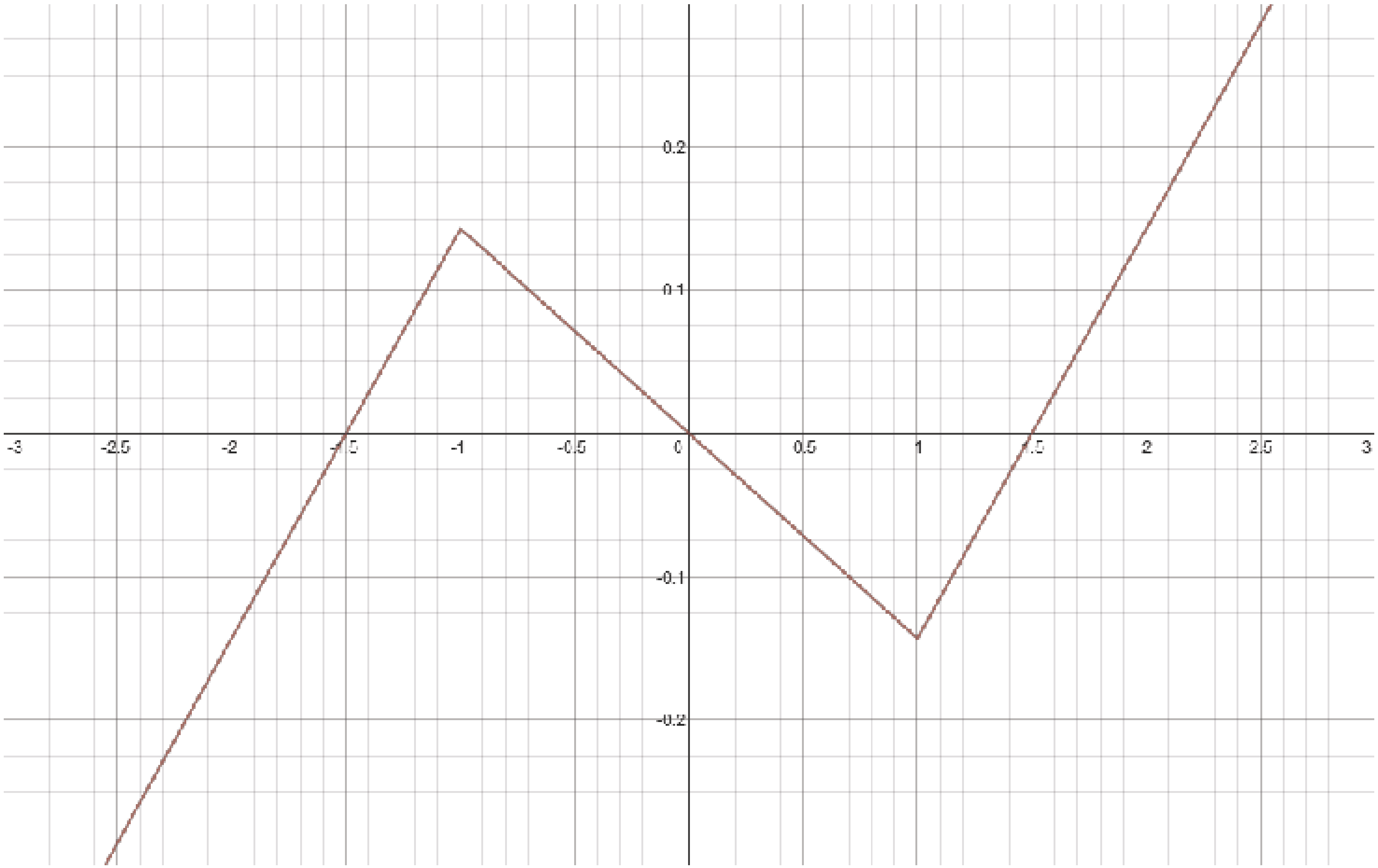
Chua Diode



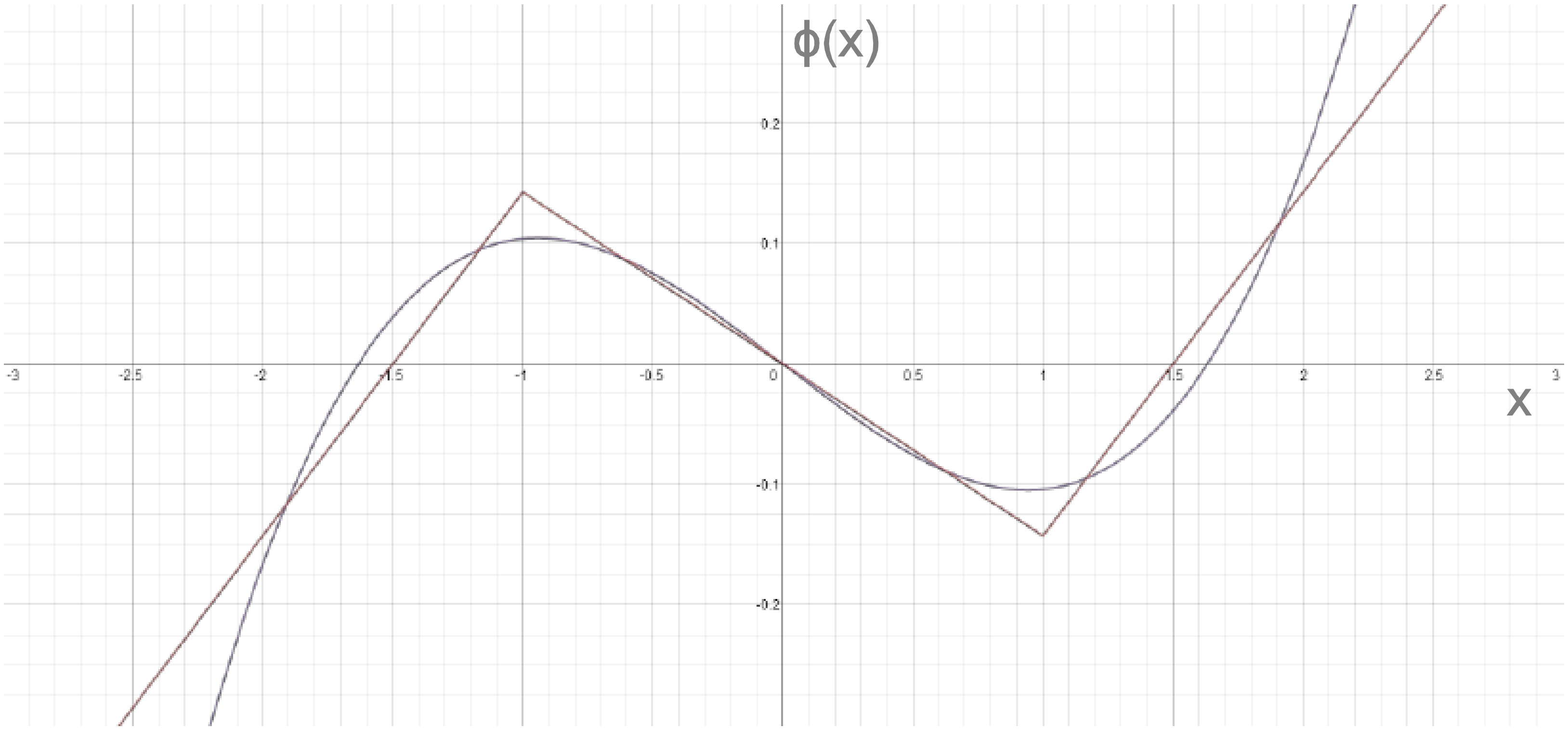


$$\phi(x) = (m_1 + 1)x + \frac{1}{2}(m_0 - m_1)[|x + 1| - |x - 1|]$$

Wait… it doesn't look nonlinear



0.2
0.1
-0.1
-0.2
-3
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
2.5
3



$$\phi(x) = \frac{1}{16}x^3 - \frac{1}{6}x$$

Found using least squares approximation of original piece-wise function

Equilibrium Points

$$x' = a(y - \phi(x))$$
$$y' = x - y - z$$
$$z' = -by$$
$$0 = a(y - (\frac{1}{16}x^3 - \frac{1}{6}x))$$
$$0 = x - y - z$$
$$0 = -by$$

$$(0,0,0), (2\sqrt{\frac{2}{3}}, 0, -2\sqrt{\frac{2}{3}}), (-2\sqrt{\frac{2}{3}}, 0, 2\sqrt{\frac{2}{3}})$$

Symmetric EP, independent of *a* and *b*

Equilibrium Points

$$\mathbf{0} = \boldsymbol{a}(\boldsymbol{y} - (\frac{\mathbf{1}}{\mathbf{16}}\boldsymbol{x}^{\mathbf{3}} - \frac{\mathbf{1}}{\mathbf{6}}\boldsymbol{x}))$$
$$\mathbf{0} = \boldsymbol{x} - \boldsymbol{y} - \boldsymbol{z}$$
$$\mathbf{0} = -\boldsymbol{b}\boldsymbol{y}$$

$$(0,0), (2\sqrt{\frac{2}{3}}, 0, -2\sqrt{\frac{2}{3}}), (-2\sqrt{\frac{2}{3}}, 0, 2\sqrt{\frac{2}{3}})$$

System is odd-symmetric. (x, y, z) replaced by $(-x, -y, -z)$ gives same null-clines. Solutions are symmetric about the origin.

Types of Solutions?

Jacobian at EP

$$\left. \begin{vmatrix} -\frac{3a}{16}x^2 + \frac{a}{6} & a & 0 \\ 1 & -1 & 1 \\ 0 & -b & 0 \end{vmatrix} \right|_{x^*}$$

where x^* is the x-value at the EP

EP at origin

We can assume $x^2 \rightarrow 0$ near origin and solution $z(t)$ decreases exponentially fast to zero. This system can be written as the following 2-D system:

$$\begin{vmatrix} -\frac{a}{6} & a \\ 1 & -1 \end{vmatrix}$$

giving eigen values of $\lambda = \frac{a}{12} - \frac{1}{2} \pm \frac{\sqrt{a^2+156a+36}}{12}$ giving real roots with opposite signs. This leads to a 3-D saddle point (unstable).

Symmetrical EP

Jacobian is the same for either EP due to the x^2 term.

$$\begin{vmatrix} -\frac{a}{3} & a & 0 \\ 1 & -1 & 1 \\ 0 & -b & 0 \end{vmatrix}$$

The characteristic equation is

$$\lambda^3 + \left(\frac{a+3}{3}\right)\lambda^2 + \left(\frac{a}{3} + b - a\right)\lambda + \frac{ab}{3}.$$

Symmetrical EP

$$\lambda^3 + \left(\frac{a+3}{3}\right)\lambda^2 + \left(\frac{a}{3} + b - a\right)\lambda + \frac{ab}{3} = 0$$

$\lambda = 0$ is not a solution since $a \neq 0$ and $b \neq 0$.

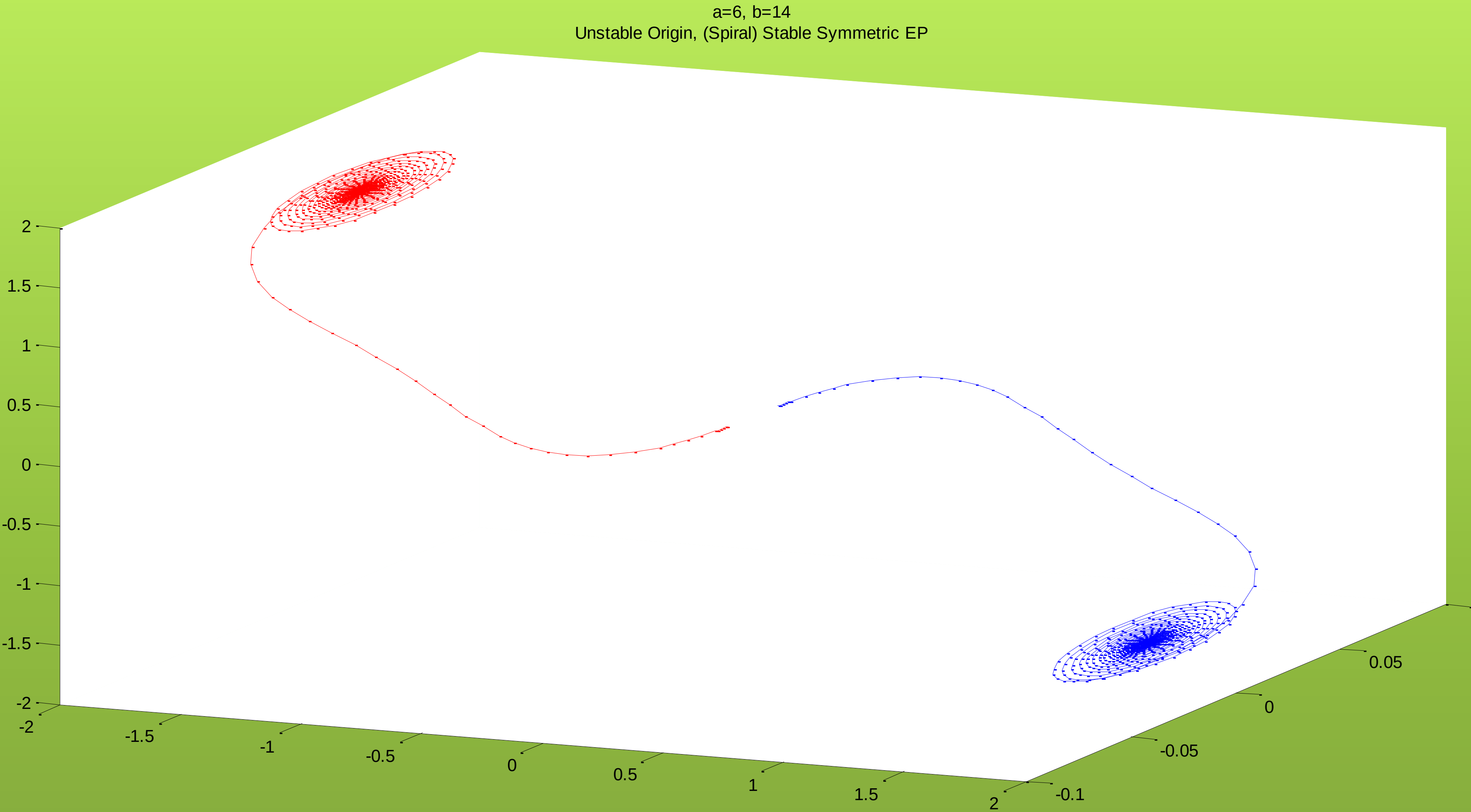
Possible solutions are saddle (real with opposite signs), a spiral source (complex with positive real portion), and a spiral sink (complex with negative real portion). Possible Hopf bifurcation when real portions go to zero, leaving pure imaginary solutions.

Set b at 14, vary a from 6 to 14

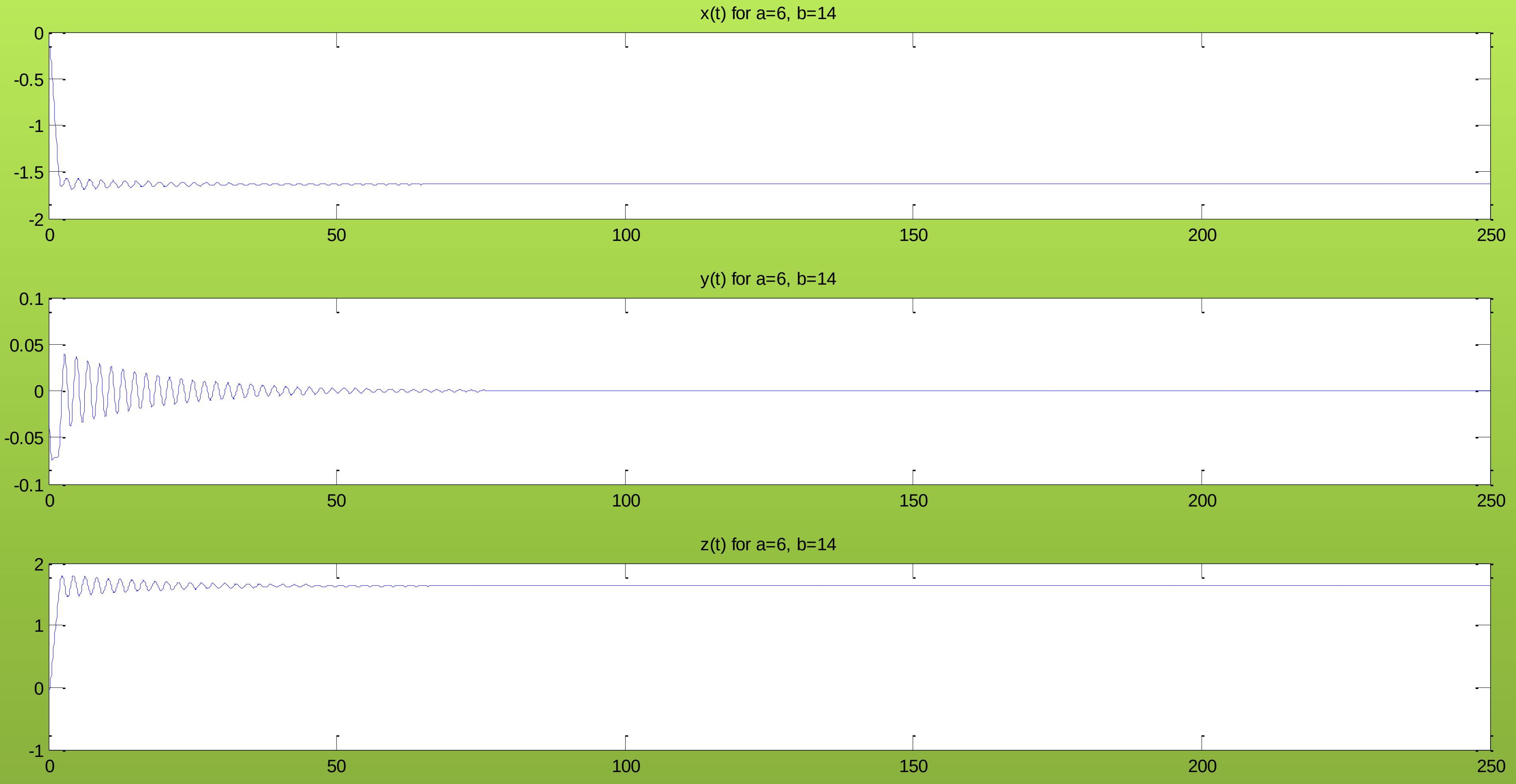
Start with $a = 6$. A pair of initial conditions are used: $(0.1,0,-0.1)$ and $(-0.1,0,0.1)$.

Two Sinks (Stable) at Symmetric EP

Solutions move to closest EP



a=6, b=14
Unstable Origin, (Spiral) Stable Symmetric EP
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
-0.1
-0.05
0
0.05


x(t) for a=6, b=14
0
-0.5
-1
-1.5
-2
0
50
100
150
200
250
y(t) for a=6, b=14
0.1
0.05
0
-0.05
-0.1
0
50
100
150
200
250
z(t) for a=6, b=14
2
1
0
-1
0
50
100
150
200
250

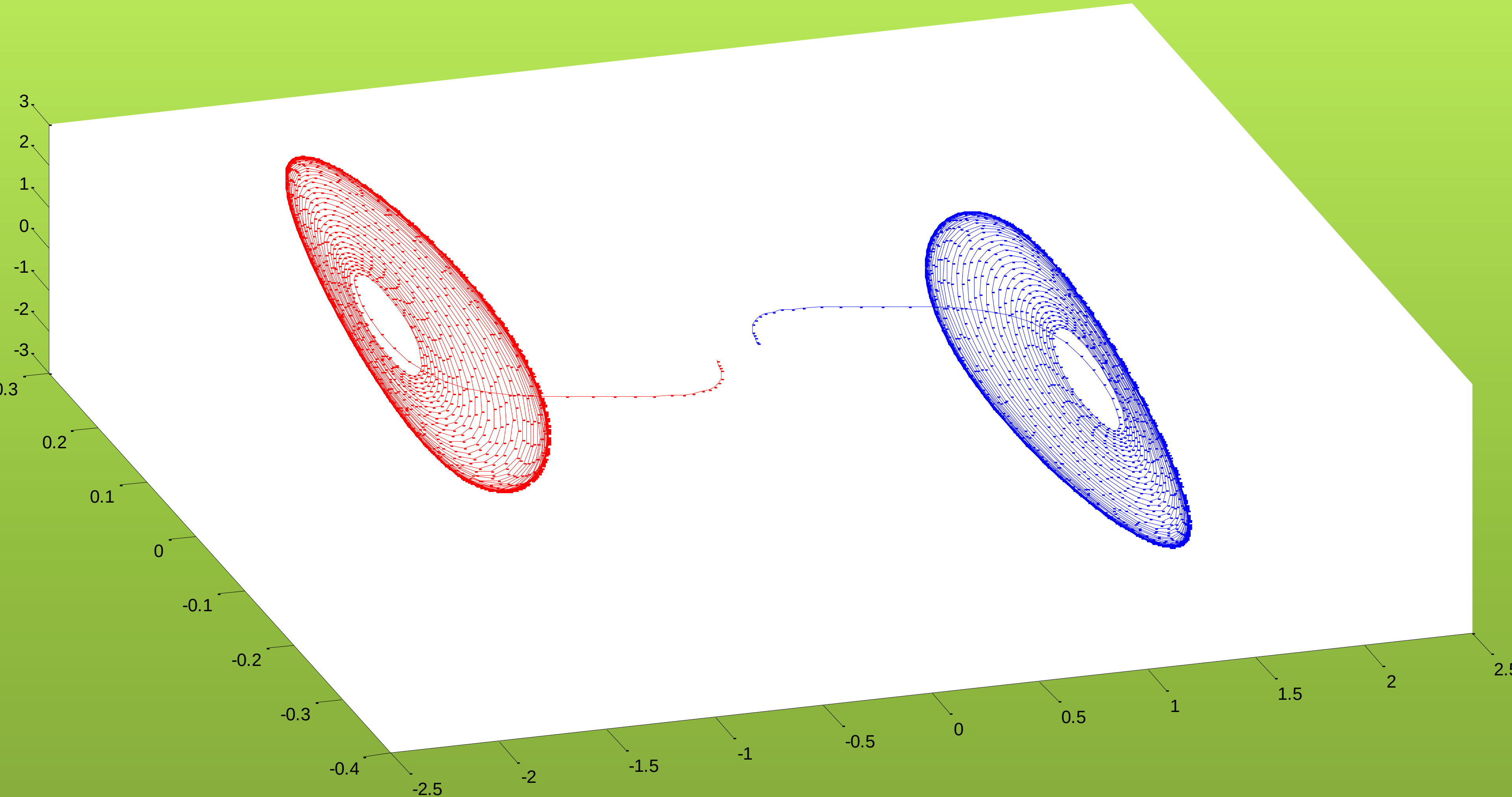
$$a = 7, b = 14$$

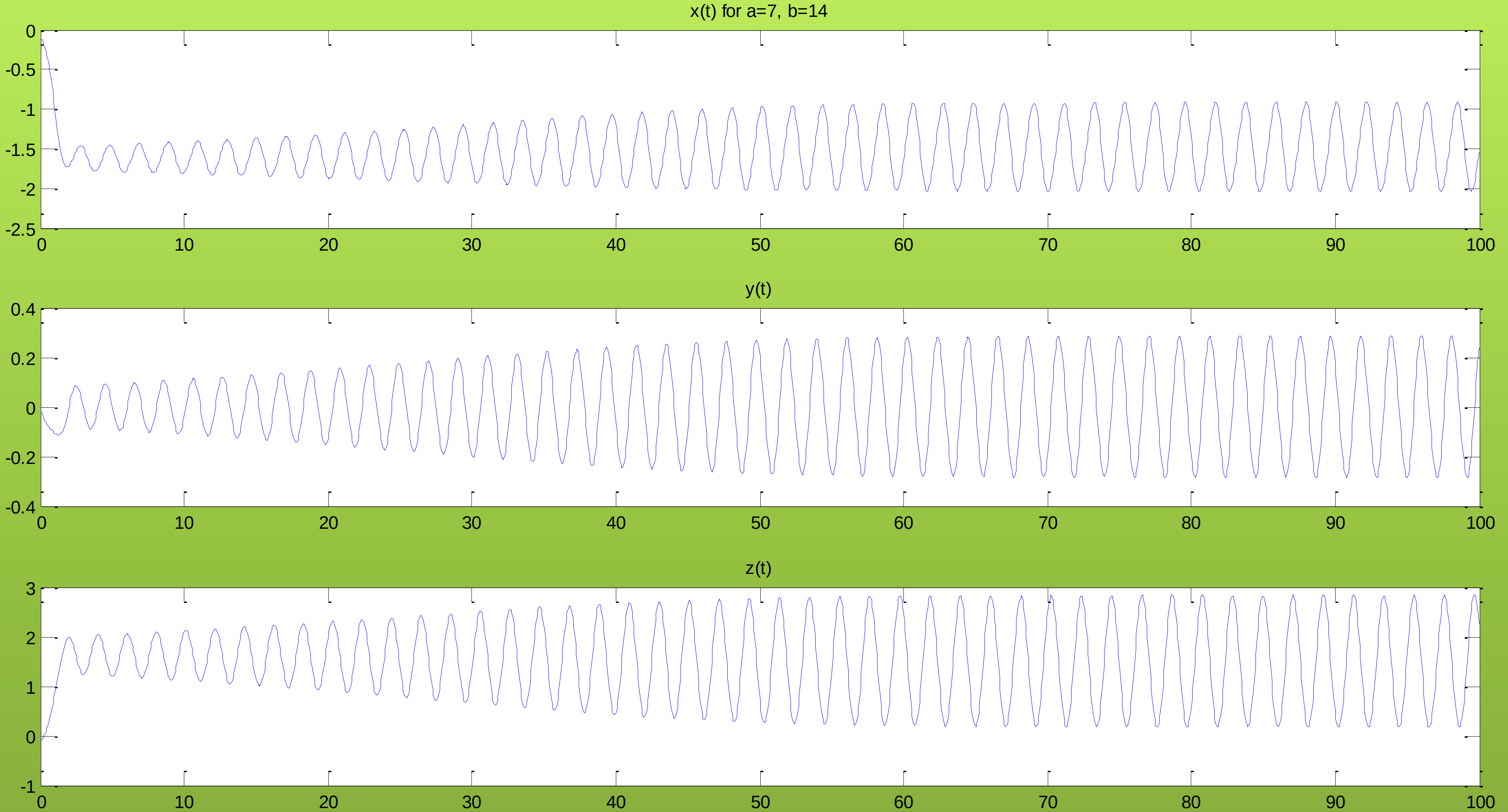
Initial conditions $(0.1, 0, -0.1)$ and $(-0.1, 0, 0.1)$.

Symmetric EPs become unstable (source) and limit cycles appear

Solutions move to closest EP

a=7, b=14
Unstable origin, Limit cycle around Symmetric EP





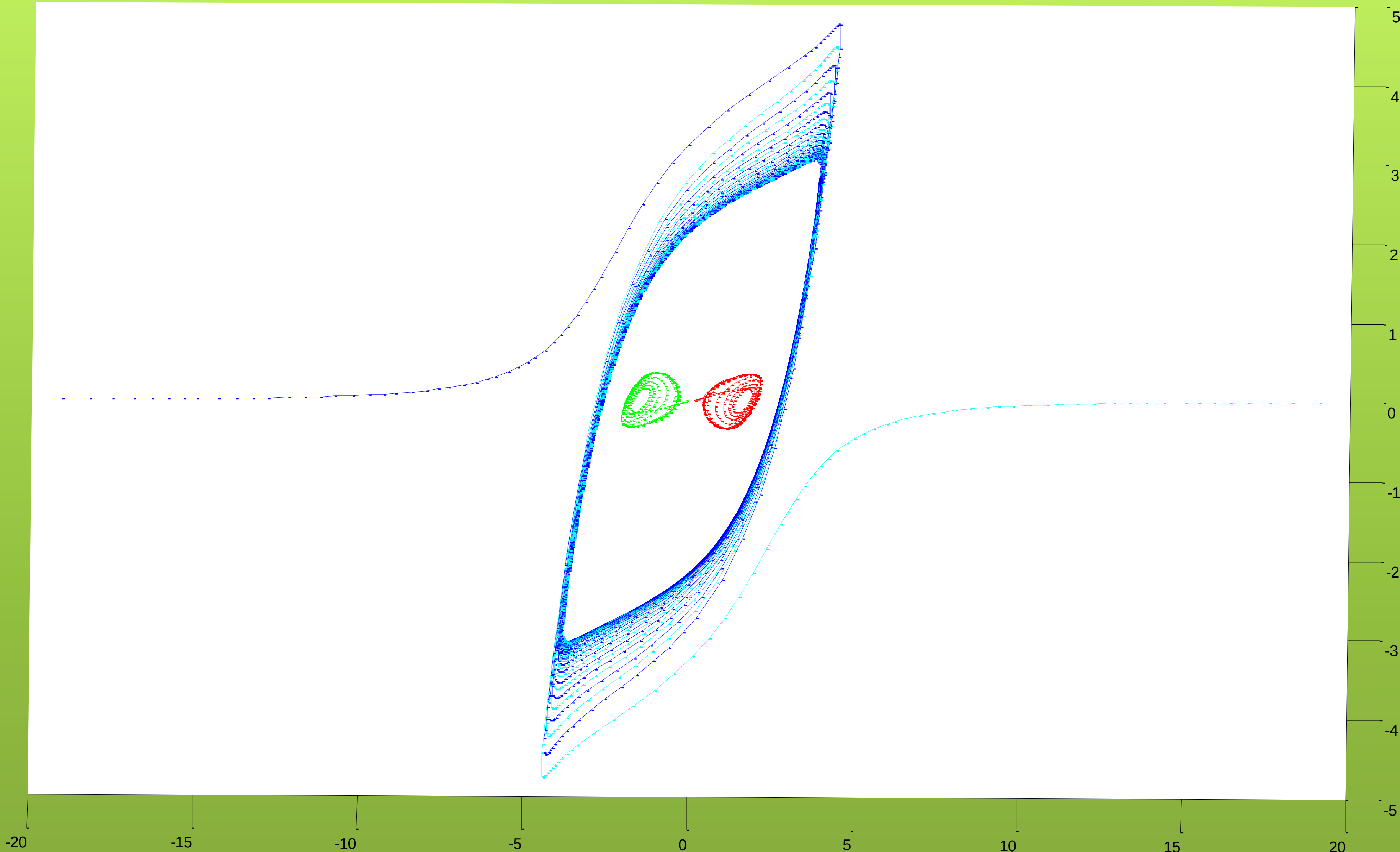
x(t) for a=7, b=14
0
-0.5
-1
-1.5
-2
-2.5
0 10 20 30 40 50 60 70 80 90 100
y(t)
0.4
0.2
0
-0.2
-0.4
0 10 20 30 40 50 60 70 80 90 100
z(t)
3
2
1
0
-1
0 10 20 30 40 50 60 70 80 90 100

$$a = 8, b = 14$$

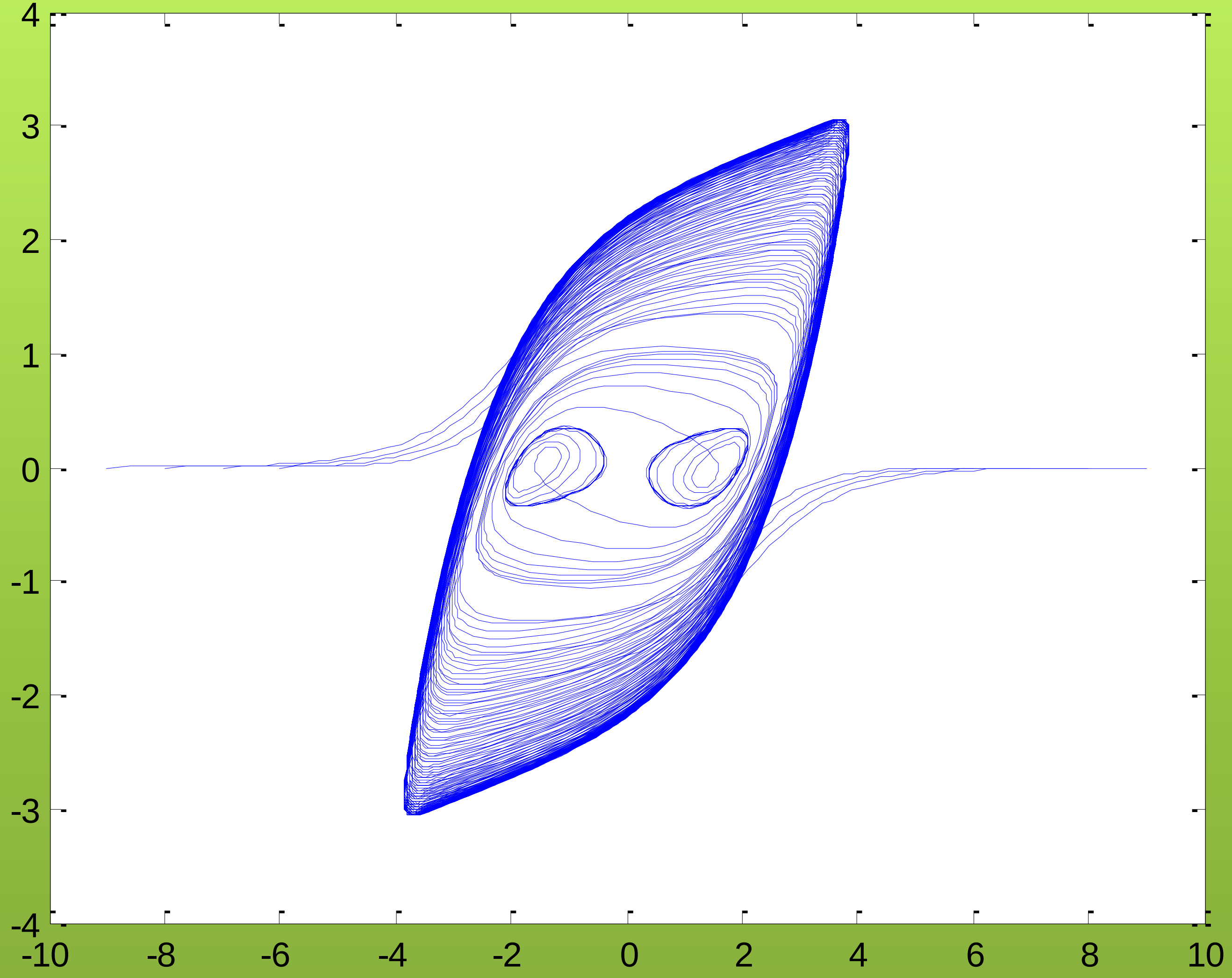
Initial conditions $(0.1,0,-0.1), (-0.1,0,0.1)$ and $(20,0,-20), (-20,0,20)$

Depending on initial conditions, solutions go towards a stable limit cycle around each EP or to a larger pair of limit cycles that encompassing both EP.

For the larger limit cycle, one limit cycle is unstable, the other is stable.



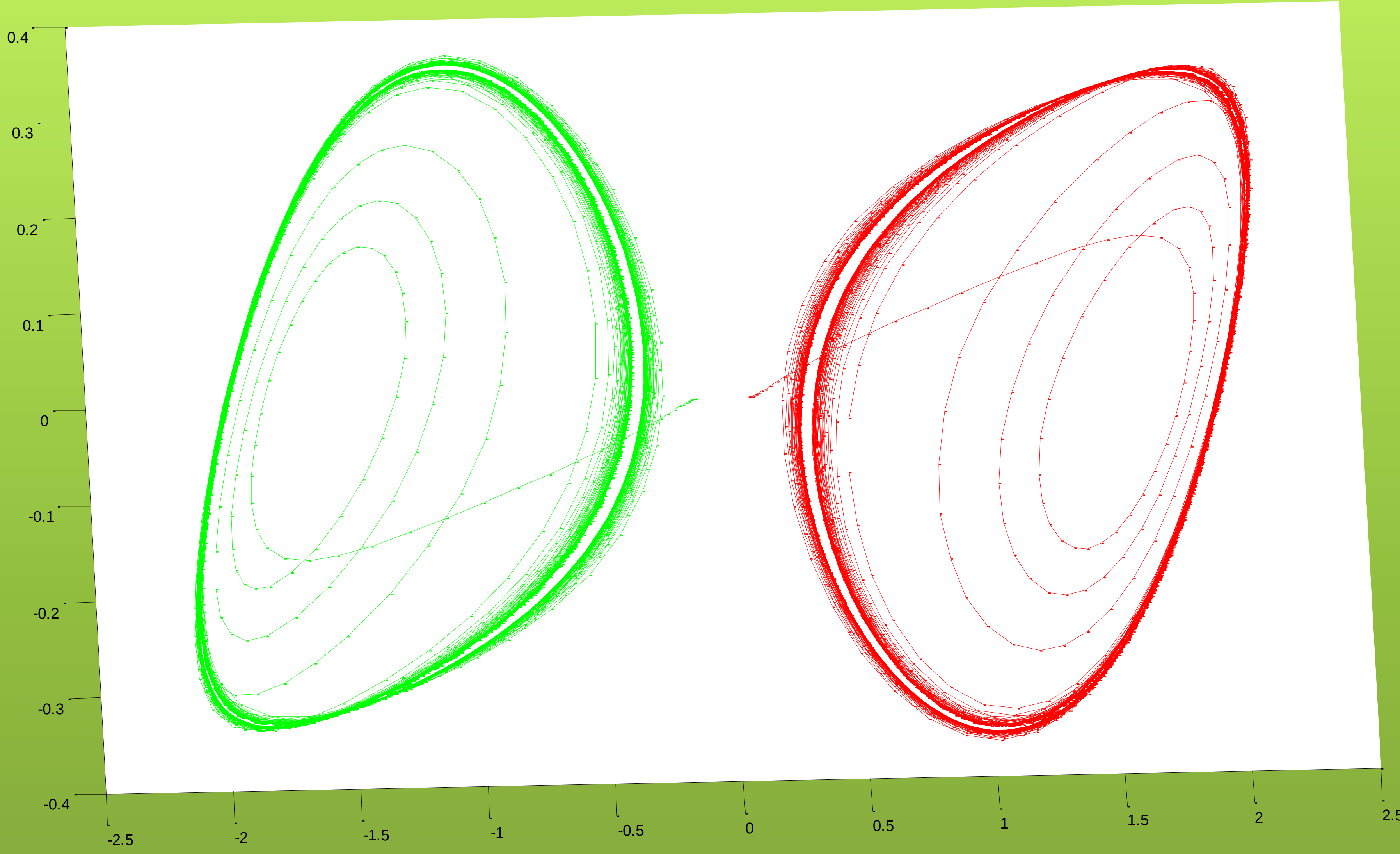
5
4
3
2
1
0
-1
-2
-3
-4
-5
-20
-15
-10
-5
0
5
10
15
20



Various initial conditions to show unstable limit cycle

$$a = 8.198, b = 14$$

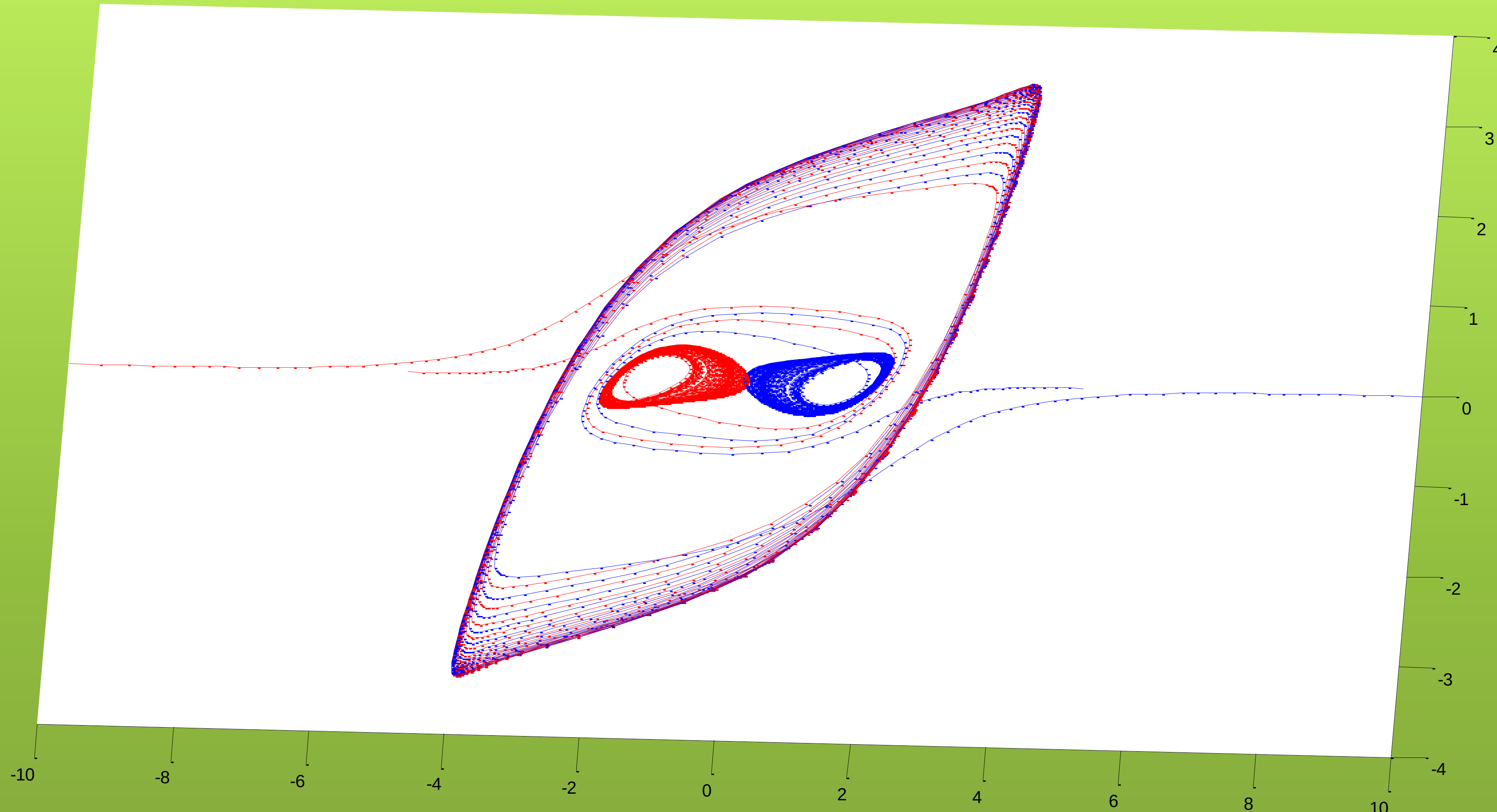
Period-doubling of small orbits around EP



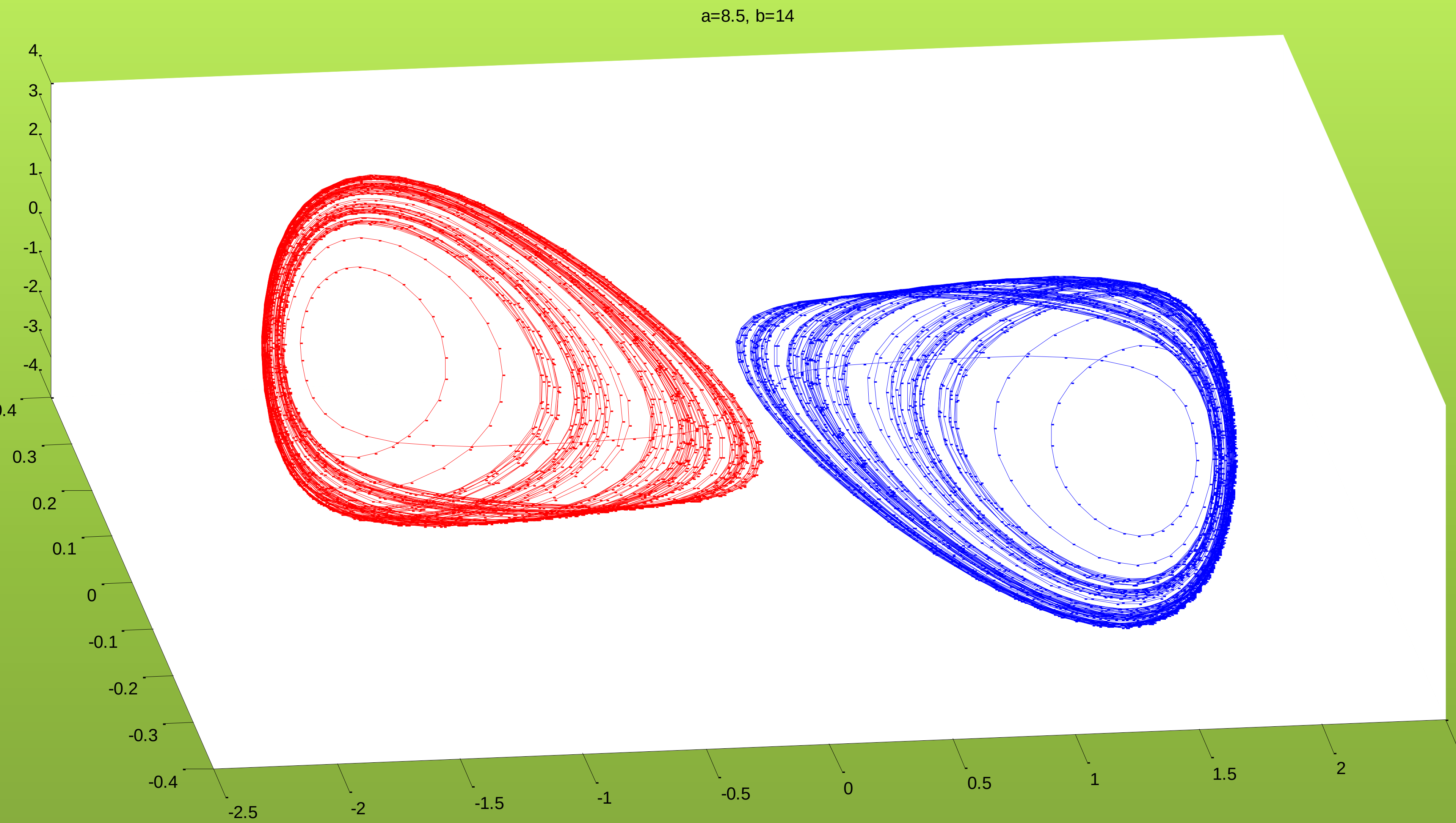
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
2.5

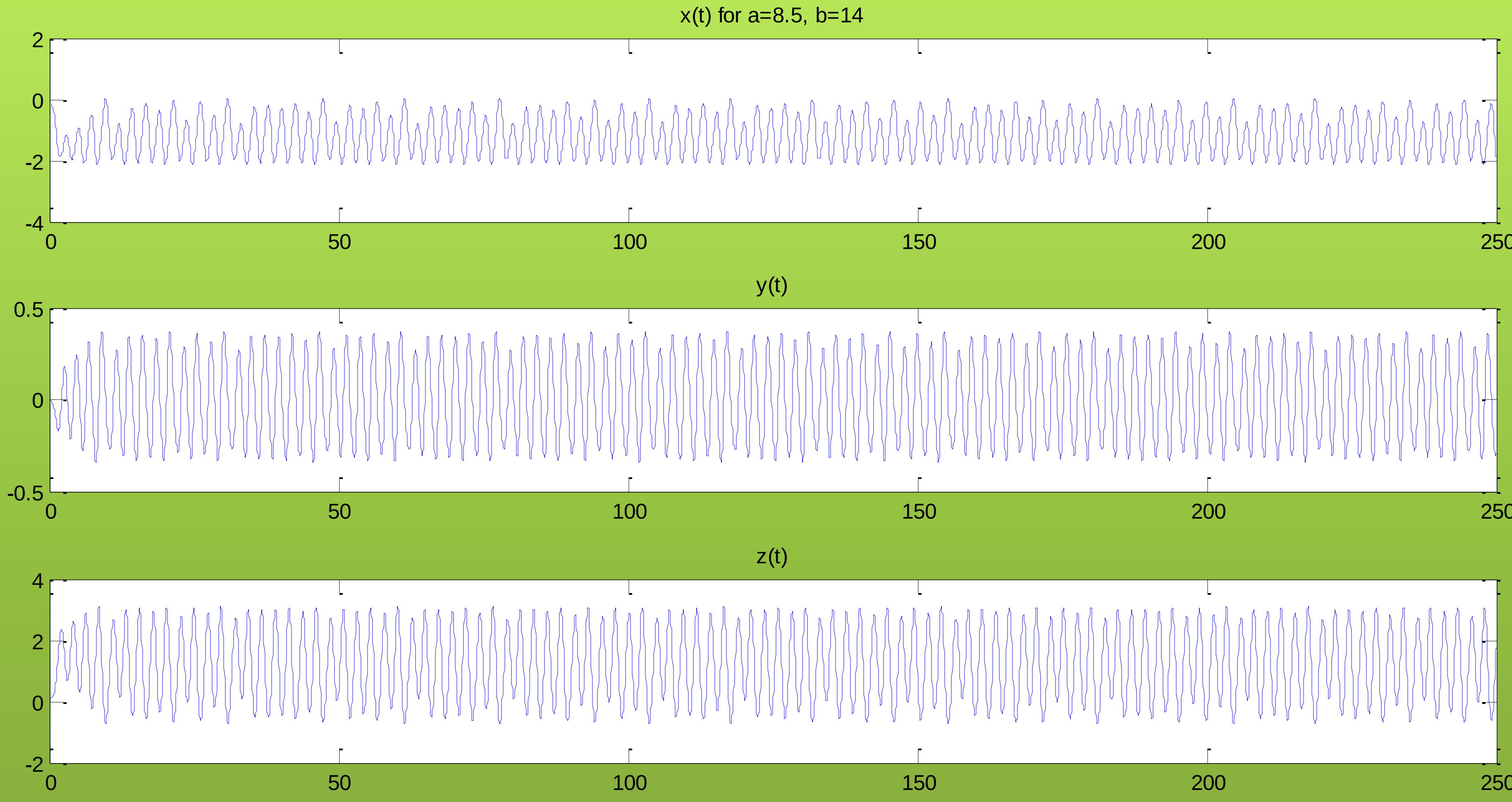
$$a = 8.5, b = 14$$

Appearance of two attractors around each EP



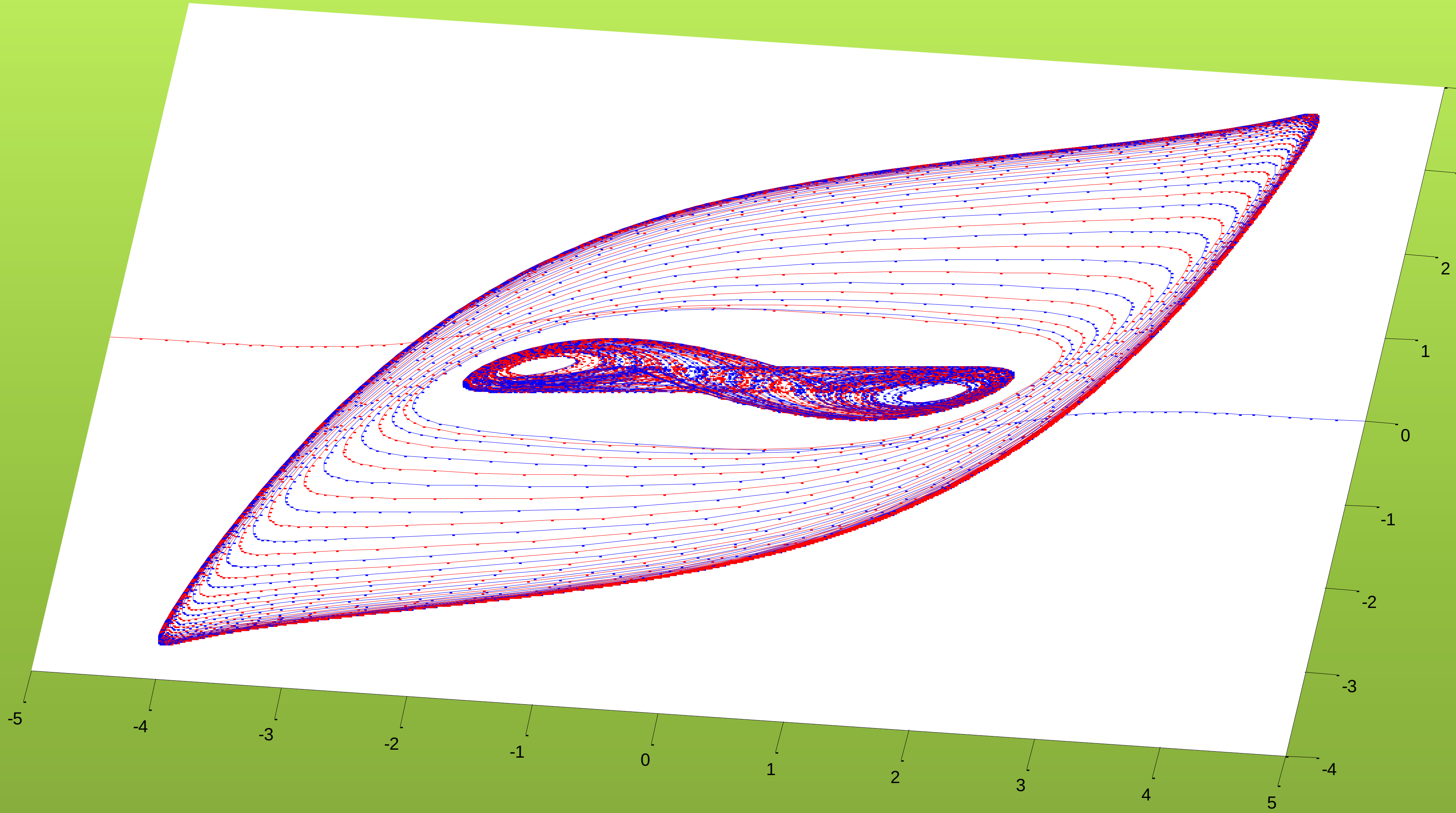
-10
-8
-6
-4
-2
0
2
4
6
8
10
-4
-3
-2
-1
0
1
2
3


a=8.5, b=14
4
3
2
1
0
-1
-2
-3
-4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2

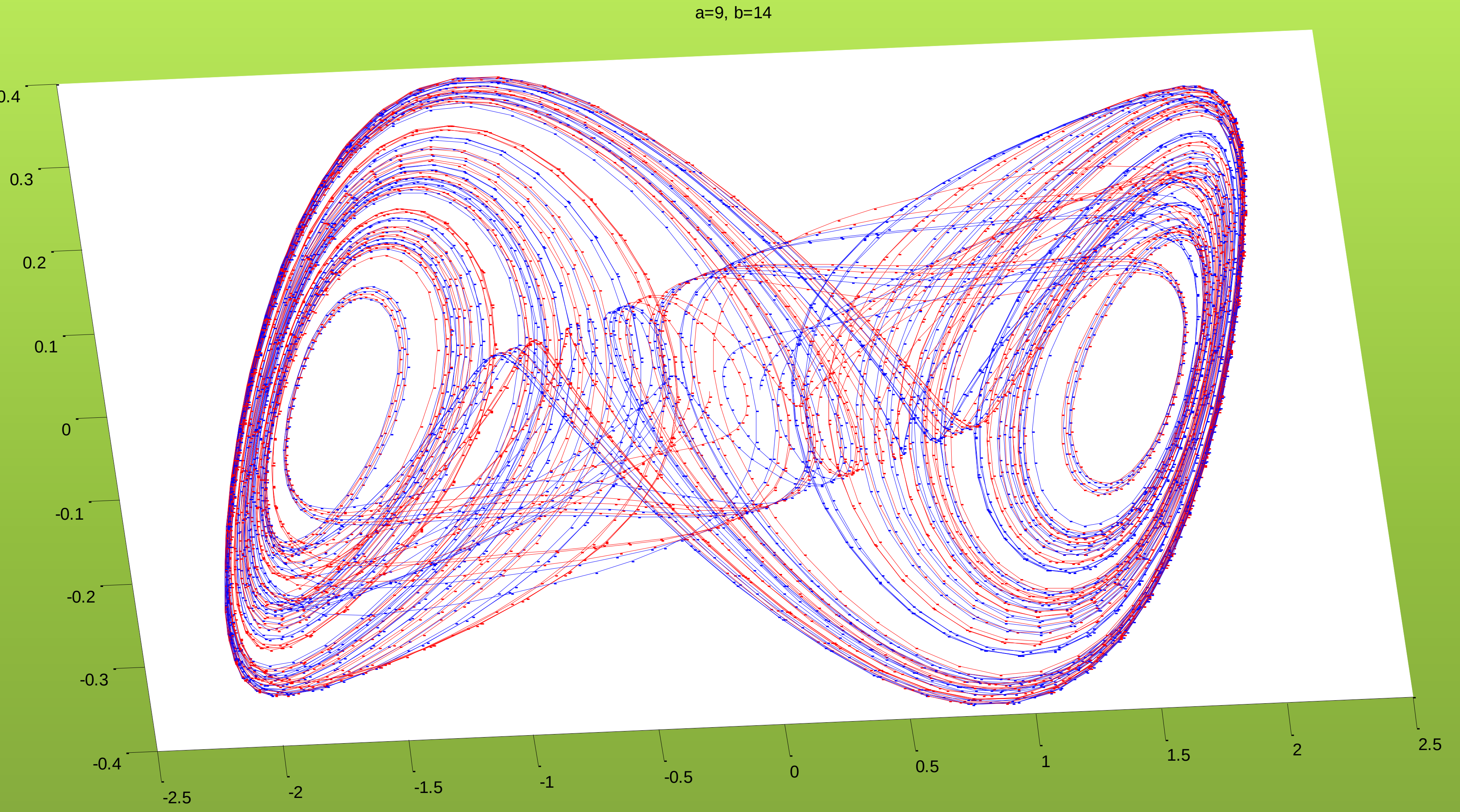

x(t) for a=8.5, b=14
2
0
-2
-4
0
50
100
150
200
250
y(t)
0.5
0
-0.5
z(t)
4
2
0
-2

$$a = 9, b = 14$$

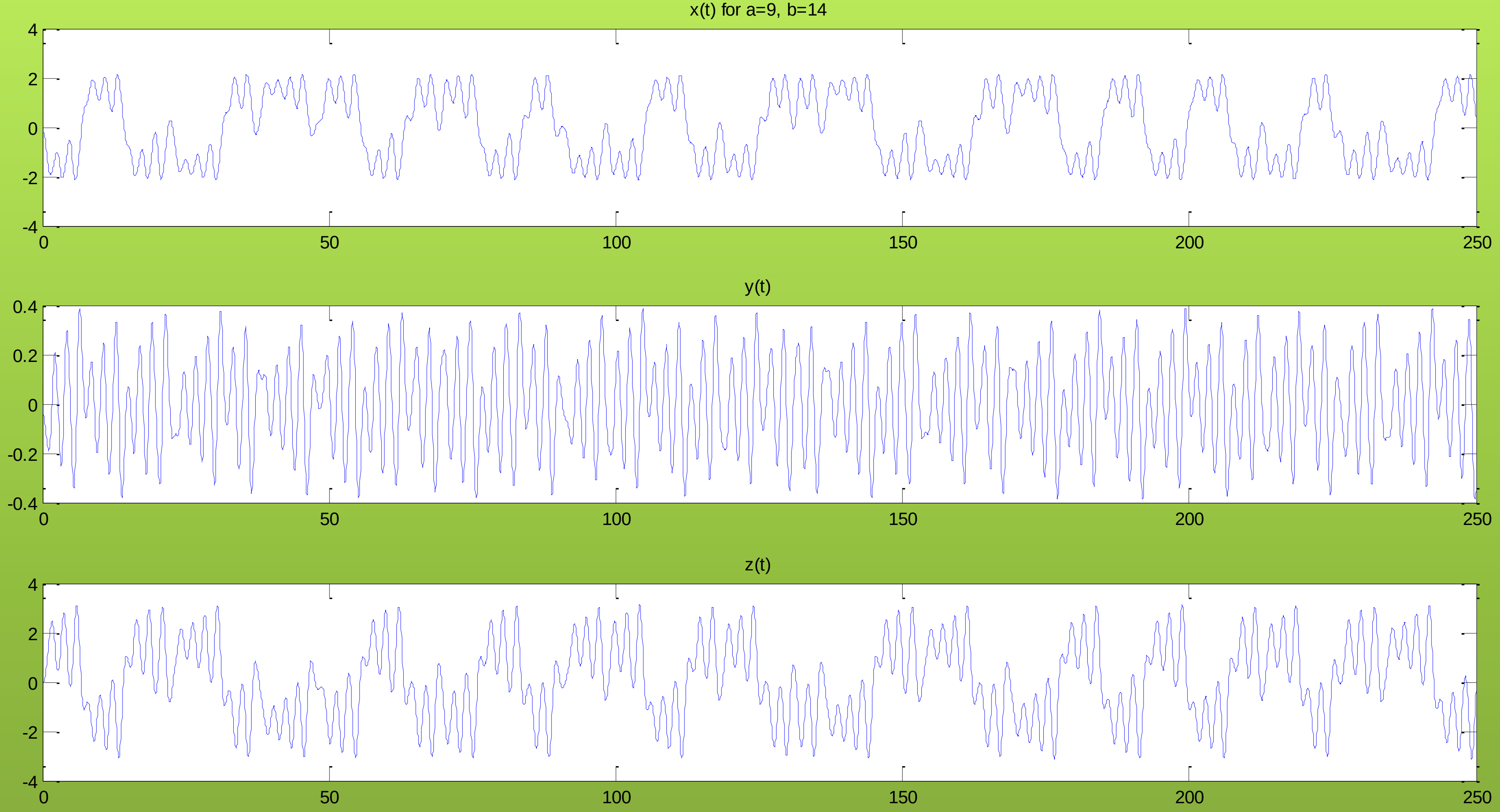
Double scroll attractor with outer periodic orbit (two attractors join together)



-5
-4
-3
-2
-1
0
1
2
3
4
5
-4
-3
-2
-1
0
1
2

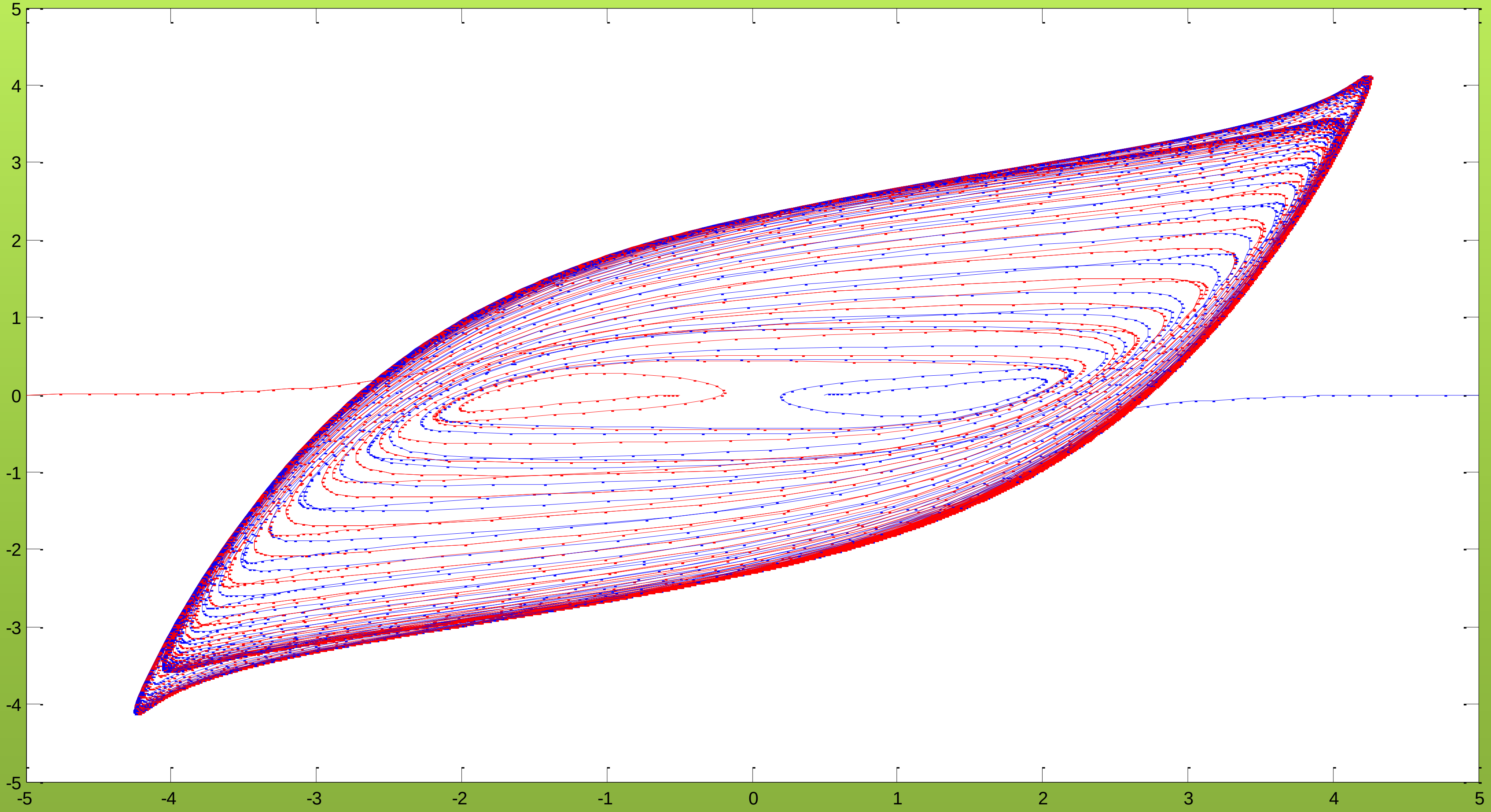


a=9, b=14
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
2.5


x(t) for a=9, b=14
y(t)
z(t)
4
2
0
-2
-4
0.4
0.2
-0.2
-0.4
50
100
150
200
250

$$a = 11, b = 14$$

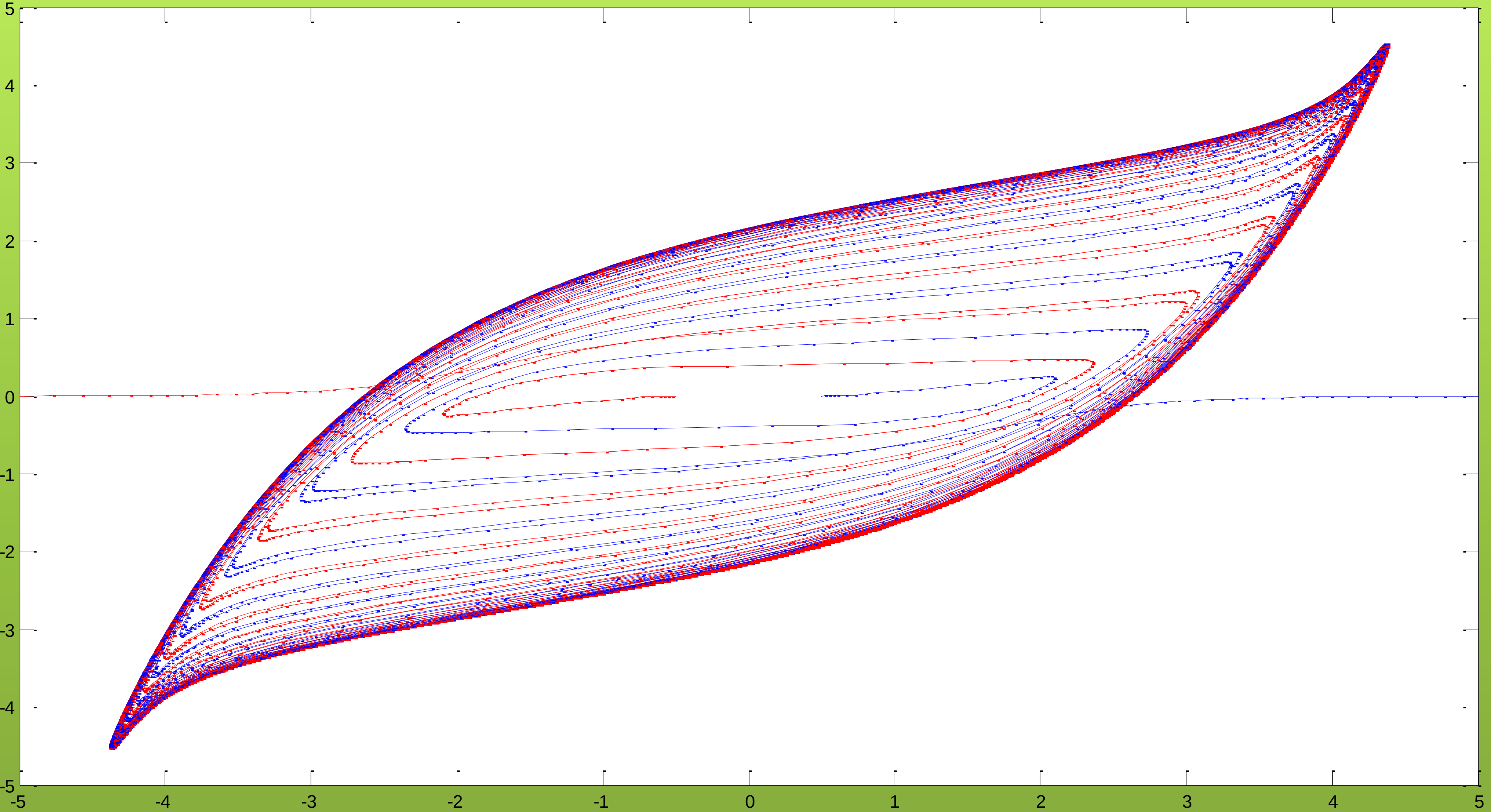
Double scroll attractor disappears – only outer periodic orbit is stable



5
4
3
2
1
0
-1
-2
-3
-4
-5
-5
-4
-3
-2
-1
0
1
2
3
4
5

$$a = 14, b = 14$$

No orbits evident around the EP – only the outer orbit exists.


5
4
3
2
1
0
-1
-2
-3
-4
-5
-5
-4
-3
-2
-1
0
1
2
3
4
5

Attractors for the system

- Single attractors appear around $a = 8.5$
- The single attractors join into a double scroll around $a = 8.785$

Bifurcations

$$\boldsymbol{a = 6.58}$$

Supercritical Andronov-Hopf Bifurcation

Symmetric EPs went from sink (stable) to source (unstable) with a stable limit cycle

$$\boldsymbol{a = 7.3}$$

Saddle-node Bifurcation of a symmetric periodic orbit

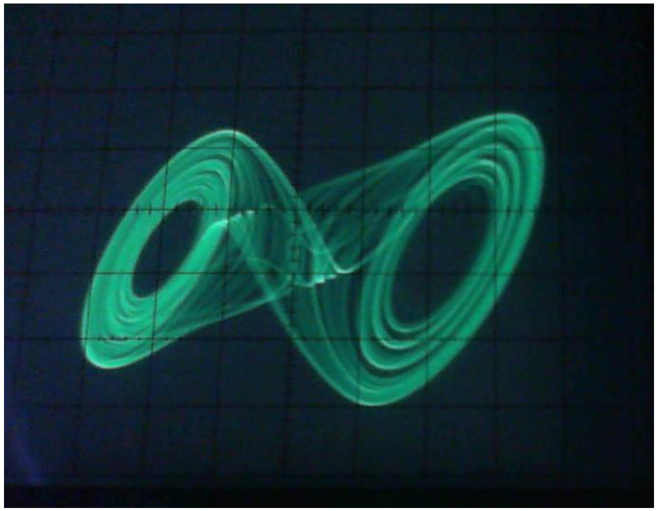
$$\boldsymbol{a = 8.78}$$

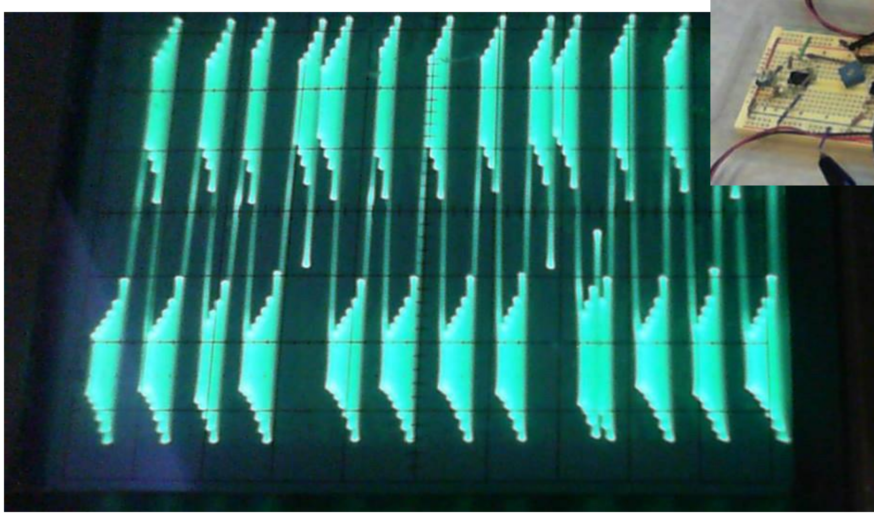
“Gluing” of asymmetric attractors to a single double scroll

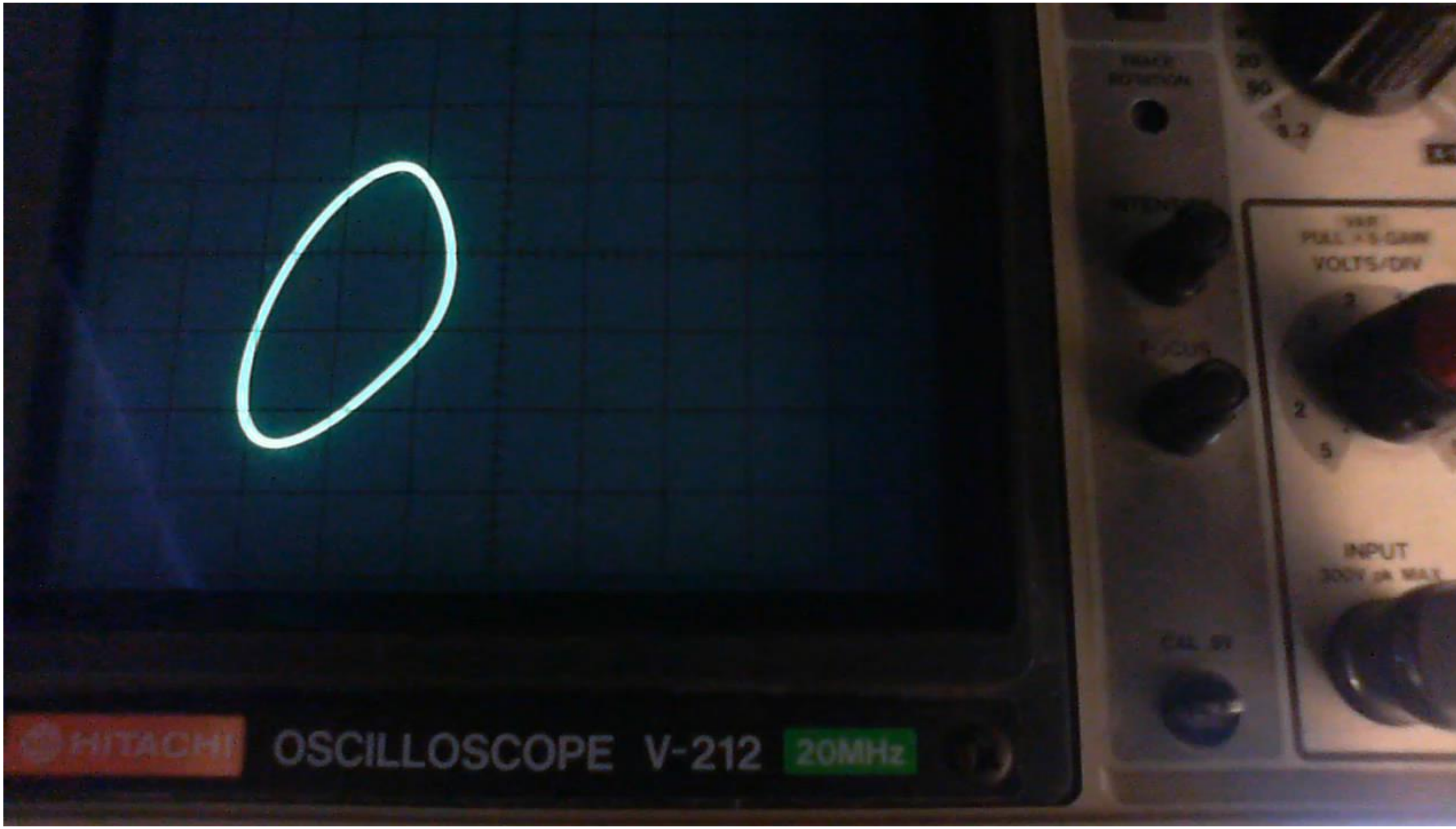
$$\boldsymbol{a = 10.77}$$

Disappearance of double scroll attractor when it collides with saddle symmetric periodic orbit

Chua Circuit in Action






VOLTS/DIV
INPUT
OSCILLOSCOPE V-212
20MHz

References

- Hirsch, M., Smale, S., Devaney, R. *Differential equations, dynamical systems, and an introduction to chaos*. Elsevier, 2013.
- Chua, L. (2007) Chua circuit. Scholarpedia, 2(10):1488.
- www.chuacircuits.com