2014 MMATYC Conference–Frederick Community College

Mathemagic with a Deck of Cards

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Abstract

There seems to be no end to the mathemagical entertainment to be had with a simple deck of cards, drawing from ideas in algebra and discrete mathematics. We'll present a sampling, both recent and classical.

The material is very much in the spirit of influential writer Martin Gardner (1914–2010), who was born 100 years ago this year, and it can be used to liven up both parties and mathematics classes.

Additional Certainties (Little Fibs)

Shuffle the deck a few times, and have two or three cards selected by different spectators.

They each remember their cards, and share the results with each other, and announce the sum of the chosen card values.

You soon announce what each individual card is!

Secret Number 1:

When several numbers are added up, each one can be determined from the sum.

Secret Number 2:

Totally free choices of cards are offered, but only from a controlled small part of the deck.

The possibilities are narrowed down by having half a dozen key cards at the top of the deck at the start, in any order, and keeping them there throughout some fair-looking shuffles.

The mathematics is about numbers (card values).

Secret Number 3:

> *You have memorized the suits of the half dozen key cards which start at the top.*

Which half dozen key cards?

We use the Fibonacci numbers:

Start with 1, 2; add to get the next one. Repeat.

$$1 + 2 = 3,$$

$$2 + 3 = 5,$$

$$3 + 5 = 8,$$

$$5 + 8 = 13,$$

and so on.

The list of Fibonacci numbers goes on forever:

1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, . . .

However, for magic with playing cards, we focus on the first six of these, namely the **Little Fibs**, i.e.,

$$1, 2, 3, 5, 8, \text{ and } 13,$$

agreeing that $1 = \text{Ace}$ and $13 = \text{King}$.

Secret Number 1 Revisited:

Little-known amazing fact:

Every positive whole number can be written in exactly one way as a sum of different *and* non-consecutive *Fibonacci numbers*

1, 2, 3, 5, 8, 13, 21, 34, ...

This was apparently not noticed before 1939, when amateur mathematician Edouard Zeckendorf from Belgium spotted it. He published it in 1972.

Zeckendorf–Fibonacci

For instance, $6 = 5 + 1$ (not $3 + 2 + 1$ or $3 + 3$), and $10 = 8 + 2$, and $20 = 13 + 5 + 2$.

Given any number, it's easy to break it up correctly: first peel off the largest possible Fib, and repeat with what's left, until we are done.

(This is a kind of greedy algorithm.)

E.g., $27 = 21 + 6 = 21 + 5 + 1$. Note, $21 = 21$.

$51 = 34 + 17 = 34 + 13 + 4 = 34 + 13 + 3 + 1$.

The Little Fibs Card Trick

Now, consider any Ace, 2, 3, 5, 8 and King, for instance.

A♣, 2♡, 3♠, 5♢, 8♣, K♡ (CHaSeD order).

If two (or three) are selected from these, then they can be determined from the sum of their values.

Any possible total can only arise in one way.

And it's easy to break up any sum into Fibs.

Generalized Fibonacci and . . . ?

The Lucas sequence 2, 1, 3, 4, 7, 11, 18, ..., which is a kind of generalized Fibonacci sequence, also has the desired property, if we omit the 2 at the start.

We don't need generalized Fibonacci sequences.

The numbers 1, 2, 4, 6, 10 work.

So do 1, 2, 5, 7, 13.

Can we generalize?

Twice as impressive?

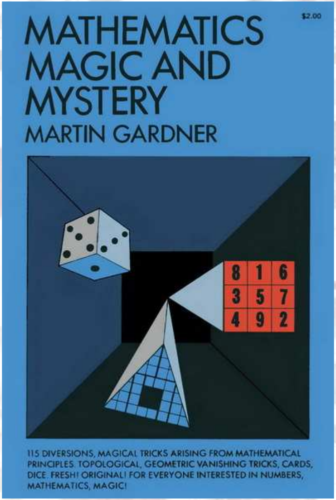
Can you have four cards picked from the six Little Fibs, the total revealed, and name all four cards?

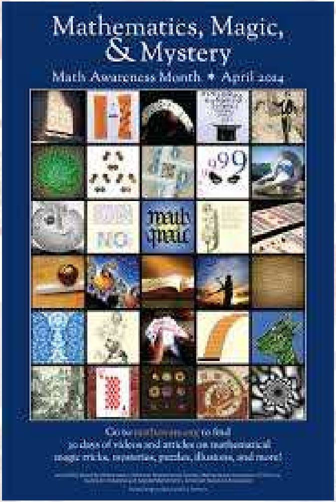
Hint: all six values add up to 32.

If four cards are chosen, and you're told the values sum to 22, then the two *not* chosen must add up to 32 - 22 $=$ 10, so *they* are the 8 and 2.

Hence the chosen cards are the Ace, 3, 5, and King.

Mathematics Awareness Month 2014!





See mathaware.org and @MathAware and Facebook.

Mathematics Awareness Month 2014

The theme is "Mathematics, Magic, and Mystery," named after a classic book by Martin Gardner.

Goal: use Martin's written legacy—over 100 books—to add to his record of turning innocent youngsters into math professors (and math professors into innocent youngsters).

There are many other Mathematics Awareness Month activities using Fibonacci numbers!

There are 30 activities overall. Have fun!

Three Scoop Miracle

Hand the deck out for shuffling. A spectator is asked to call out her favourite ice-cream flavour; let's suppose she says, "Chocolate."

Take the cards back, and take off about a quarter of the deck. Mix them further until told when to stop.

Deal cards, one for each letter of "chocolate," before dropping the rest on top as a topping.

This spelling/topping routine is repeated twice more—so three times total.

Three Scoop Miracle

Emphasize how random the dealing was, since the cards were shuffled and you had no control over the named ice-cream flavour.

Have the spectator press down hard on the card that ends up on top, requesting that she magically turn it into a specfic card, say the four of diamonds.

When that card is turned over it will indeed be found to be the desired card.

Low Down Triple Dealing

The key move here is a *reversed transfer* of a fixed number of cards in a packet—at least half—from top to bottom, done three times total.

The dealing out of k cards from a packet that runs $\{1, 2, \ldots, k-1, k, k+1, k+2, \ldots, n-1, n\}$ from the top down, and then dropping the rest on top as a unit, yields the rearranged packet

$$\{k+1, k+2, \ldots, n-1, n, k, k-1, \ldots, 2, 1\}.$$

All true, but hardly inspiring!

Low Down Triple Dealing

When $k \geq \frac{n}{2}$, doing this three times brings the bottom card(s) to the top. Why?

Given a flavour of length k and a number $n \leq 2k$, the packet of size n breaks symmetrically into three pieces T, M, B of sizes $n-k$, $2k-n$, $n-k$, such that the count-out-and-transfer operation (of k cards each time) is

$$T, M, B \to B, \overline{M}, \overline{T},$$

where the bar indicates complete reversal.

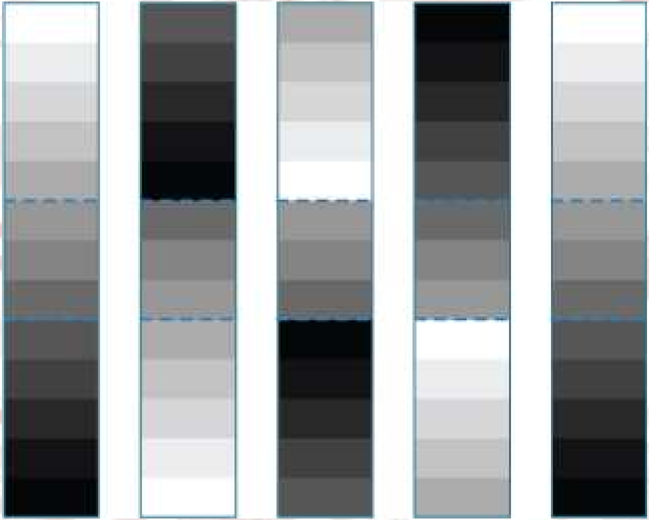
Low Down Triple Dealing Generalized?

Using this approach, the Bottom to Top (with three moves) property can be proved. Actually . . .

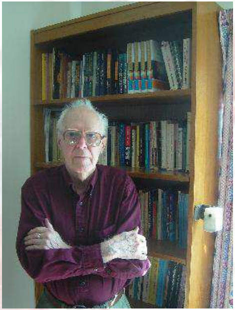
The Bottom to Top property is only 75% of the story. Here's the real scoop:

The Period 4 Principle
If four reversed transfers of k cards are done to a packet of size n, where $k \geq \frac{n}{2}$, then every card in the packet is returned to its original position.

A picture is worth . . .



Credit where credit is due:



Martin Gardner (1914-2010)
The Best Friend Mathematics Ever Had
–standing by every word he ever wrote

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www.martin-gardner.org

Given Any Five Cards

A spectator selects five random cards from a deck and hands them to a volunteer from the audience.

The volunteer glances at the cards, and hands one back to the spectator, who hides it.

The volunteer places the remaining four cards in a face-up row on the table.

The mathemagician glances at the cards on the table, and after a suitable pause, promptly reveals the identity of the hidden card.

Fitch Cheney's Five-Card Trick

This is really a two-person card trick, although the second key person—the volunteer—is actually a mathematical accomplice of the mathemagician, who has been trained in advance!

How can this be done without verbal or physical cues?

This effect is due to mathematician William Fitch Cheney Jnr (1904-1974), who in 1927 received the first PhD in Mathematics awarded by MIT.

Fitch Cheney's Five-Card Trick

Note that *the volunteer* gets to choose which card to hand back, and then, in what order to place the remaining four cards.

Three main ideas make this magic possible:

1. At least two cards are of the same suit.

WLOG the volunteer has two Clubs. One Club is handed back/hidden, and by placing the remaining four cards in some order, the volunteer effectively communicated the identity of that hidden Club.

2. The volunteer can use one designated position (e.g., the first) of the four available for the retained Club—which determines the suit of the hidden card

The volunteer uses the other three positions for the placement of the remaining cards, which can be arranged in $3! = 6$ ways.

If the volunteer and mathemagician agree on a one-to-one correspondence between the six possible permutations of those three cards and the numbers $1, 2, \ldots, 6$, then the volunteer can communicate one of six things to the mathemagician.

Fitch Cheney's Five-Card Trick

What *can* one say about these other three cards? Not much—for instance, some or all of them could be Clubs too, or there could be other suit matches!

However, one thing *is* certain: they are all distinct, so with respect to some total ordering of the entire deck, one of them is LOW, one is MEDIUM, and one is HIGH. Asssume deck is in CHaSeD order.

Using permutations, any number between 1 and 6 can be communicated.

Fitch Cheney's Five-Card Trick

But surely 6 isn't enough? After all, the hidden card could in general be any one of 12 Clubs!

This brings us to the third main idea:

3. The volunteer must be careful as to which card he hands back to the spectator.

Considering the 13 card values, Ace, $2, 3, \ldots, 10$, J, Q, K, as being arranged clockwise on a circle, we see that the two suit match cards are at most 6 values apart, i.e., counting clockwise, one of them lies at most 6 vertices past the other.

Fitch Cheney's Five-Card Trick

The volunteer gives this "higher" valued Club back to the spectator, and uses the "lower" Club, along with the other three cards, to communicate the identity of the hidden card.

For example, if the volunteer has the 2♣ and 8♣, then he hands back the 8♣. But if he has the 2♣ and J♣, he hands back the 2♣.

In general, he saves one card of a particular suit and needs to communicate another of the same suit, whose numerical value is k higher than the one he displays, for some k between 1 and 6 inclusive.

Fitch Cheney's Five-Card Trick

Put this total CHaSeD ordering on the whole deck:

$$\begin{array}{l} A\clubsuit,\ 2\clubsuit,\ \dots\ ,\ K\clubsuit, \\ A\heartsuit,\ 2\heartsuit,\ \dots\ ,\ K\heartsuit, \\ A\spadesuit,\ 2\spadesuit,\ \dots\ ,\ K\spadesuit, \\ A\diamondsuit,\ 2\diamondsuit,\ \dots\ ,\ K\diamondsuit. \end{array}$$

Mentally, he labels the three cards L (low), M (medium), and H (high) w.r.t. this ordering.

The 6 permutations of L,M,H are always ordered by rank, i.e., 1 = LMH, 2 = LHM, 3 = MLH, 4 = MHL, 5 = HLM and 6 = HML.

Finally, he orders the three cards in the pile from left to right according to this scheme to communicate the desired integer.

For example, if he is playing the J♣ and trying to communicate the 2♣ to the mathemaician, then $k = 4$, and he plays the other three cards in the order MHL.

The mathemagician knows that the hidden card is a Club, decodes the MHL as 4, and mentally counts 4 past the visible J♣ (mod 13) to get the 2♣.

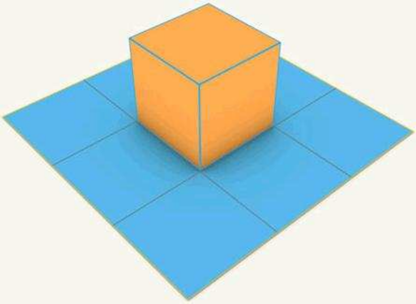
Fitch Four Glory

A spectator from the crowd chooses any four cards at random and hands them to the volunteer. He glances at them briefly, and hands one back, which the spectator then places face down to one side.

The volunteer quickly place the remaining three cards in a row on the table, some face up, some face down, from left to right.

The mathemagician, who has not seen anything so far, now looks at the cards on display, and promptly names the hidden fourth card—even in the case where all three cards are face down!

Let's wrap it up



Wrap the gold cube completely with the blue paper!
All cutting and folding must be along grid lines.

The paper must remain in one piece.